Physics of Information Technology Unit 9 Antennas

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Problem (10.1): Electric Field for an Infinitesimal Dipole Radiator

Goal: Derive the far-field electric field produced by an infinitesimal (Hertz) dipole.

Step-by-Step Explanation:

Step 1: Start with the Vector Potential: According to the document (Eq. (9.25)), the vector potential for an infinitesimal dipole (of length d and current amplitude I_0) is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I_0 d\, e^{-ikr}}{4\pi r}\,\hat{z}\,,$$

where $k = \omega/c$ and $r = |\mathbf{r}|$. (This equation is derived by integrating the current distribution over the small length of the dipole.)

Step 2: Express A in Spherical Coordinates: In spherical coordinates, the unit vector $\hat{z}$ can be written as

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\,. \quad \text{(see Eq. (9.26))}$$

Therefore, the vector potential becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I_0 d\, e^{-ikr}}{4\pi r}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right].$$

Step 3: Find the Magnetic Field B: The magnetic field is obtained by taking the curl of $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \quad \text{(see Eq. (9.27))}$$

Detailed derivation (using spherical coordinates and the curl formula in Eq. (7.21)) yields a dominant component in the ϕ direction. The document provides the result as

$$B_\phi = \frac{\mu_0 I_0 d}{4\pi} e^{-ikr}\left(\frac{ik}{r} + \frac{1}{r^2}\right)\sin\theta\,. \quad \text{(Eq. (9.28))}$$

Step 4: Derive the Electric Field E: In the radiation zone (far-field), the electric field is related to the magnetic field by a phase shift and the intrinsic impedance of free space. Maxwell's equations lead to the relation

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}\,. \quad \text{(Eq. (9.3))}$$

In the far-field, the dominant term comes from the time derivative of $\mathbf{A}$. Taking a time derivative (with e^{-ikr} representing a time dependence $e^{-i\omega t}$) gives

$$\mathbf{E} \approx j\omega\mathbf{A}\,.$$

Focusing on the transverse (i.e. θ) component and neglecting higher-order terms (the $1/r^2$ term), we obtain

$$E_\theta \approx \frac{j\omega\mu_0 I_0 d}{4\pi r} e^{-ikr}\sin\theta\,. \quad \text{(Derived from Eq. (9.29))}$$

Here, $j\omega\mathbf{A}$ indicates that the field oscillates with time and the factor j shows a 90° phase shift relative to $\mathbf{A}$.

Summary: The far-field electric field (dominant θ-component) of an infinitesimal dipole is

$$\boxed{E_\theta(r,\theta) \approx \frac{j\omega\mu_0 I_0 d}{4\pi r} e^{-ikr} \sin\theta\,.}$$

This result is obtained by starting with the vector potential (Eq. (9.25)), converting it into spherical coordinates (Eq. (9.26)), taking the curl to find $\mathbf{B}$ (Eq. (9.28)), and then using Maxwell's equations to find $\mathbf{E}$ (as in Eq. (9.29)).

Problem (10.2): Poynting Vector and Field Strength at 1 km

Question: For an isotropic radiator emitting 1 kW of power, determine:

- The magnitude of the Poynting vector at a distance of 1 km.
- The peak electric field strength at that distance.

Step-by-Step Explanation:

Step 1: Power Density: An isotropic radiator emits equally in all directions. Therefore, at a distance r, the power density is

$$S = \frac{P}{4\pi r^2}\,.$$

With $P = 1000\,\mathrm{W}$ and $r = 1000\,\mathrm{m}$, we have

$$S = \frac{1000}{4\pi(1000)^2} \approx \frac{1000}{12.57 \times 10^6} \approx 7.96 \times 10^{-5}\,\mathrm{W/m^2}\,. \quad \text{(Derived from basic spherical spreading)}$$

Step 2: Relating Poynting Vector to Electric Field: For a plane wave in free space, the time-average Poynting vector is given by

$$\langle S \rangle = \frac{E_0^2}{2\eta}\,,$$

where E_0 is the peak electric field and $\eta \approx 377\,\Omega$ is the intrinsic impedance of free space. (This comes from the relation $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ and $H = E/\eta$.)

Step 3: Solve for E_0: Rearranging,

$$E_0 = \sqrt{2\eta\langle S \rangle}\,.$$

Substitute $\eta = 377\,\Omega$ and $\langle S \rangle \approx 7.96 \times 10^{-5}\,\mathrm{W/m^2}$:

$$E_0 = \sqrt{2 \times 377 \times 7.96 \times 10^{-5}}\,\mathrm{V/m}\,.$$

Compute the product: $2 \times 377 \approx 754$, then $754 \times 7.96 \times 10^{-5} \approx 0.06$. Taking the square root,

$$E_0 \approx \sqrt{0.06} \approx 0.245\,\mathrm{V/m}\,.$$

Summary: The magnitude of the power density at 1 km is about $7.96 \times 10^{-5}\,\mathrm{W/m^2}$, and the peak electric field strength is

$$\boxed{E_0 \approx 0.245\,\mathrm{V/m}\,.}$$

These results use spherical spreading (basic physics) and the relation between the Poynting vector and the electric field.

Problem (10.3): Maximum Power Delivery and Load Resistance

Question: In the effective antenna circuit (Figure 9.3 of the document), for what value of R_{load} is the maximum power delivered to the load?

Step-by-Step Explanation:

Step 1: Maximum Power Transfer Theorem: The maximum power transfer theorem states that maximum power is delivered to the load when the load resistance equals the source (antenna) resistance (in the case of purely resistive impedances). (This is a basic circuit theorem found in any introductory circuit textbook.)

Step 2: Application: For the antenna circuit, if we denote the antenna's radiation resistance as R_{rad}, then maximum power is delivered when

$$R_{\text{load}} = R_{\text{rad}}\,.$$

Summary:

$$\boxed{R_{\text{load}} = R_{\text{rad}}\,. \quad \text{(Maximum Power Transfer Condition)}}$$

Problem (10.4): Gain, Effective Area, and Their Ratio for an Infinitesimal Dipole

Question: For an infinitesimal dipole antenna, determine the gain and the effective area, and show that their ratio is $\lambda^2/(4\pi)$.

Step-by-Step Explanation:

Step 1: Gain of an Infinitesimal Dipole: It is a standard result in antenna theory (see Eq. (9.48) and related discussion) that the maximum gain G of a Hertz dipole is approximately 1.5 (in linear terms). (This result comes from comparing the maximum Poynting vector with that of an isotropic radiator.)

Step 2: Effective Area: The effective aperture (area) A of an antenna is related to its gain by the formula

$$A = \frac{\lambda^2 G}{4\pi}\,.$$

(This is a well-known result in antenna theory and is mentioned in the text following Eq. (9.52).)

Step 3: Ratio of Effective Area to Gain: Dividing A by G, we obtain

$$\frac{A}{G} = \frac{\lambda^2}{4\pi}\,.$$

Summary: For an infinitesimal dipole antenna, the gain is approximately 1.5, and its effective area is

$$A = \frac{\lambda^2 G}{4\pi}\,.$$

Thus, the ratio is

$$\boxed{\frac{A}{G} = \frac{\lambda^2}{4\pi}\,.}$$

Conclusion and Historical Context

These problems use a mix of electromagnetic theory and circuit analysis:

- Problem (10.1) starts from Maxwell's equations and the definition of the vector potential (Eq. (9.25)) to derive the radiated electric field of a dipole.
- Problem (10.2) uses the concept of spherical spreading and the relation between the Poynting vector and the electric field (Eq. (9.30) and the standard $S = E^2/(2\eta)$ formula).
- Problem (10.3) applies the maximum power transfer theorem from basic circuit theory.
- Problem (10.4) links antenna gain and effective aperture through a fundamental relation in antenna theory.

These derivations are rooted in 19th-century discoveries by Maxwell, Hertz, and others, and they form the basis of modern antenna design and wireless communications.